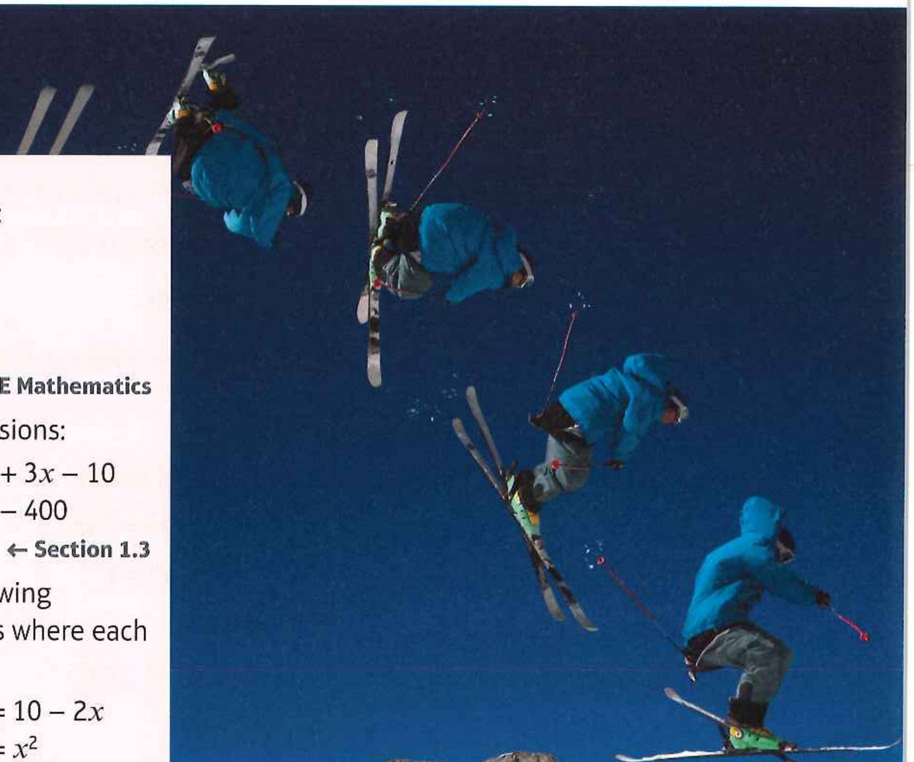
2 Quadratics

Objectives

After completing this chapter you should be able to:

- Solve quadratic equations using factorisation, the quadratic formula and completing the square → pages 19 – 24
- Read and use f(x) notation when working with functions → pages 25 – 27
- Sketch the graph and find the turning point of a quadratic function → pages 27 – 30
- Find and interpret the discriminant of a quadratic expression → pages 30 – 32
- Use and apply models that involve quadratic functions → pages 32 – 35



Prior knowledge check

1 Solve the following equations:

a $3x + 6 = x - 4$

b $5(x + 3) = 6(2x - 1)$

c $4x^2 = 100$

d $(x - 8)^2 = 64$ ← GCSE Mathematics

2 Factorise the following expressions:

a $x^2 + 8x + 15$ **b** $x^2 + 3x - 10$

c $3x^2 - 14x - 5$ **d** $x^2 - 400$

← Section 1.3

3 Sketch the graphs of the following equations, labelling the points where each graph crosses the axes:

a $y = 3x - 6$ **b** $y = 10 - 2x$

c $x + 2y = 18$ **d** $y = x^2$

← GCSE Mathematics

4 Solve the following inequalities:

a $x + 8 < 11$ **b** $2x - 5 \geqslant 13$

c $4x - 7 \leqslant 2(x - 1)$ **d** $4 - x < 11$

← GCSE Mathematics

Quadratic functions are used to model **projectile motion**. Whenever an object is thrown or launched, its path will approximately follow the shape of a **parabola**. → Mixed exercise Q11

2.1 Solving quadratic equations

A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where a, b and c are real constants, and $a \neq 0$. Quadratic equations can have one, two, or no real solutions.

- **To solve a quadratic equation by factorising:**
  - **Write the equation in the form $ax^2 + bx + c = 0$**
  - **Factorise the left-hand side**
  - **Set each factor equal to zero and solve to find the value(s) of x**

Notation The solutions to an equation are sometimes called the **roots** of the equation.

Example 1

Solve the following equations:

a $x^2 - 2x - 15 = 0$ **b** $x^2 = 9x$

c $6x^2 + 13x - 5 = 0$ **d** $x^2 - 5x + 18 = 2 + 3x$

a $x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$ — Factorise the quadratic. ← Section 1.3

Then either $x + 3 = 0 \Rightarrow x = -3$ — If the product of the factors is zero, one of the factors must be zero.

or $x - 5 = 0 \Rightarrow x = 5$

So $x = -3$ and $x = 5$ are the two solutions of the equation. — A quadratic equation with two distinct factors has two distinct solutions.

Notation The symbol $\Rightarrow$ means 'implies that'. This statement says 'If $x + 3 = 0$, then $x = -3$'.

Watch out The signs of the solutions are **opposite** to the signs of the constant terms in each factor.

b $x^2 = 9x$

$x^2 - 9x = 0$ — Be careful not to divide both sides by x, since x may have the value 0. Instead, rearrange into the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$ — Factorise.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

The solutions are $x = 0$ and $x = 9$.

c $6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$ — Factorise.

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$ — Solutions to quadratic equations do not have to be integers. The quadratic equation $(px + q)(rx + s) = 0$ will have solutions $x = -\frac{q}{p}$ and $x = -\frac{s}{r}$.

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ and $x = -\frac{5}{2}$

d $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$ — Rearrange into the form $ax^2 + bx + c = 0$.

$(x - 4)(x - 4) = 0$ — Factorise.

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Notation When a quadratic equation has exactly one root it is called a **repeated root**. You can also say that the equation has two equal roots.

In some cases it may be more straightforward to solve a quadratic equation without factorising.

Example 2

Solve the following equations

a $(2x-3)^2 = 25$ **b** $(x-3)^2 = 7$

a $(2x-3)^2 = 25$

$2x - 3 = \pm 5$ — Take the square root of both sides. Remember $5^2 = (-5)^2 = 25$.

$2x = 3 \pm 5$ — Add 3 to both sides.

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ and $x = -1$

b $(x-3)^2 = 7$ — Take square roots of both sides.

$x - 3 = \pm\sqrt{7}$

$x = 3 \pm \sqrt{7}$

The solutions are $x = 3 + \sqrt{7}$ and

$x = 3 - \sqrt{7}$ — You can leave your answer in surd form.

Notation The symbol ± lets you write two statements in one line of working. You say 'plus or minus'.

Exercise 2A

1 Solve the following equations using factorisation:

a $x^2 + 3x + 2 = 0$ **b** $x^2 + 5x + 4 = 0$ **c** $x^2 + 7x + 10 = 0$ **d** $x^2 - x - 6 = 0$

e $x^2 - 8x + 15 = 0$ **f** $x^2 - 9x + 20 = 0$ **g** $x^2 - 5x - 6 = 0$ **h** $x^2 - 4x - 12 = 0$

2 Solve the following equations using factorisation:

a $x^2 = 4x$ **b** $x^2 = 25x$ **c** $3x^2 = 6x$ **d** $5x^2 = 30x$

e $2x^2 + 7x + 3 = 0$ **f** $6x^2 - 7x - 3 = 0$ **g** $6x^2 - 5x - 6 = 0$ **h** $4x^2 - 16x + 15 = 0$

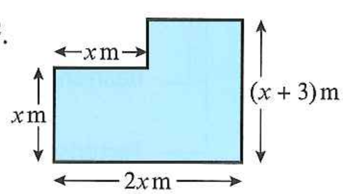
3 Solve the following equations:

a $3x^2 + 5x = 2$ **b** $(2x-3)^2 = 9$ **c** $(x-7)^2 = 36$ **d** $2x^2 = 8$ **e** $3x^2 = 5$

f $(x-3)^2 = 13$ **g** $(3x-1)^2 = 11$ **h** $5x^2 - 10x^2 = -7 + x + x^2$

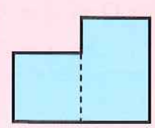
i $6x^2 - 7 = 11x$ **j** $4x^2 + 17x = 6x - 2x^2$

(P) 4 This shape has an area of $44\,\text{m}^2$.
Find the value of x.



Problem-solving

Divide the shape into two sections:



(P) 5 Solve the equation $5x + 3 = \sqrt{3x + 7}$.

Some equations cannot be easily factorised. You can also solve quadratic equations using the **quadratic formula**.

- **The solutions of the equation $ax^2 + bx + c = 0$ are given by the formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Watch out You need to rearrange the equation into the form $ax^2 + bx + c = 0$ before reading off the coefficients.

Example 3

Solve $3x^2 - 7x - 1 = 0$ by using the formula.

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-1)}}{2 \times 3}$$

$a = 3$, $b = -7$ and $c = -1$.

Put brackets around any negative values.

$$x = \frac{7 \pm \sqrt{49 + 12}}{6}$$

$-4 \times 3 \times (-1) = +12$

$$x = \frac{7 \pm \sqrt{61}}{6}$$

Then $x = \frac{7 + \sqrt{61}}{6}$ or $x = \frac{7 - \sqrt{61}}{6}$

Or $x = 2.47$ (3 s.f.) or $x = -0.135$ (3 s.f.)

Exercise 2B

1 Solve the following equations using the quadratic formula.
Give your answers exactly, leaving them in surd form where necessary.

a $x^2 + 3x + 1 = 0$ **b** $x^2 - 3x - 2 = 0$ **c** $x^2 + 6x + 6 = 0$ **d** $x^2 - 5x - 2 = 0$
e $3x^2 + 10x - 2 = 0$ **f** $4x^2 - 4x - 1 = 0$ **g** $4x^2 - 7x = 2$ **h** $11x^2 + 2x - 7 = 0$

2 Solve the following equations using the quadratic formula.
Give your answers to three significant figures.

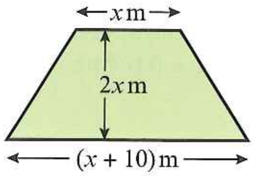
a $x^2 + 4x + 2 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $x^2 + 11x - 9 = 0$ **d** $x^2 - 7x - 17 = 0$
e $5x^2 + 9x - 1 = 0$ **f** $2x^2 - 3x - 18 = 0$ **g** $3x^2 + 8 = 16x$ **h** $2x^2 + 11x = 5x^2 - 18$

3 For each of the equations below, choose a suitable method and find all of the solutions.
Where necessary, give your answers to three significant figures.

a $x^2 + 8x + 12 = 0$ **b** $x^2 + 9x - 11 = 0$
c $x^2 - 9x - 1 = 0$ **d** $2x^2 + 5x + 2 = 0$
e $(2x + 8)^2 = 100$ **f** $6x^2 + 6 = 12x$
g $2x^2 - 11 = 7x$ **h** $x = \sqrt{8x - 15}$

Hint You can use any method you are confident with to solve these equations.

(P) **4** This trapezium has an area of $50\,\text{m}^2$.
Show that the height of the trapezium is equal to $5(\sqrt{5} - 1)\,\text{m}$.



Problem-solving
Height must be positive. You will have to discard the negative solution of your quadratic equation.

Challenge
Given that x is positive, solve the equation
$$\frac{1}{x} + \frac{1}{x+2} = \frac{28}{195}$$

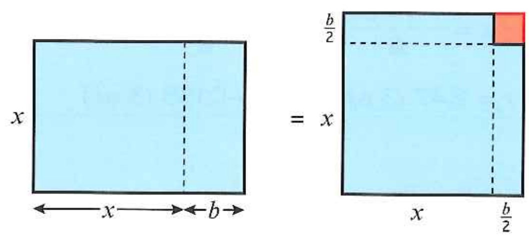
Hint Write the equation in the form $ax^2 + bx + c = 0$ before using the quadratic formula or factorising.

2.2 Completing the square

It is frequently useful to rewrite quadratic expressions by **completing the square:**

- $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

You can draw a diagram of this process when x and b are positive:



$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

The original rectangle has been rearranged into the shape of a square with a smaller square missing. The two areas shaded blue are the same.

Example 4

Complete the square for the expressions:

a $x^2 + 8x$ **b** $x^2 - 3x$ **c** $2x^2 + 12x$

a $x^2 + 8x = (x + 4)^2 - 4^2$
$= (x + 4)^2 - 16$

Begin by halving the coefficient of x. Using the rule given above, $b = 8$ so $\frac{b}{2} = 4$.

b $x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$
$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$

Be careful if $\frac{b}{2}$ is a fraction. Here $\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$.

c $2x^2 - 12x = 2(x^2 - 6x)$
$= 2((x - 3)^2 - 3^2)$
$= 2((x - 3)^2 - 9)$
$= 2(x - 3)^2 - 18$

Here the coefficient of x^2 is 2, so take out a factor of 2. The other factor is in the form $(x^2 + bx)$ so you can use the rule to complete the square.

Expand the outer bracket by multiplying 2 by 9 to get your answer in this form.

Notation A quadratic expression in the form $p(x + q)^2 + r$ where p, q and r are real constants is in **completed square form**.

- $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a^2}\right)$

Example 5

Write $3x^2 + 6x + 1$ in the form $p(x + q)^2 + r$, where p, q and r are integers to be found.

$3x^2 + 6x + 1$
$= 3(x^2 + 2x) + 1$
$= 3((x + 1)^2 - 1^2) + 1$
$= 3(x + 1)^2 - 3 + 1$
$= 3(x + 1)^2 - 2$
So $p = 3$, $q = 1$ and $r = -2$.

Watch out This is an **expression**, so you can't divide every term by 3 without changing its value. Instead, you need to take a factor of 3 out of $3x^2 + 6x$.

You could also use the rule given above to complete the square for this expression, but it is safer to learn the method shown here.

Exercise 2C

Hint In question **3d**, write the expression as $-4x^2 - 16x + 10$ then take a factor of -4 out of the first two terms to get $-4(x^2 + 4x) + 10$.

1 Complete the square for the expressions:

a $x^2 + 4x$ **b** $x^2 - 6x$ **c** $x^2 - 16x$ **d** $x^2 + x$ **e** $x^2 - 14$

2 Complete the square for the expressions:

a $2x^2 + 16x$ **b** $3x^2 - 24x$ **c** $5x^2 + 20x$ **d** $2x^2 - 5x$ **e** $8x - 2x^2$

3 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:

a $2x^2 + 8x + 1$ **b** $5x^2 - 15x + 3$ **c** $3x^2 + 2x - 1$ **d** $10 - 16x - 4x^2$ **e** $2x - 8x^2 + 10$

(E) **4** Given that $x^2 + 3x + 6 = (x + a)^2 + b$, find the values of the constants a and b. **(2 marks)**

(E) **5** Write $2 + 0.8x - 0.04x^2$ in the form $A - B(x + C)^2$, where A, B and C are constants to be determined. **(3 marks)**

Example 6

Solve the equation $x^2 + 8x + 10 = 0$ by completing the square.
Give your answers in surd form.

$x^2 + 8x + 10 = 0$ — Check coefficient of $x^2 = 1$.
$x^2 + 8x = -10$ — Subtract 10 to get the LHS in the form $x^2 + bx$.
$(x + 4)^2 - 4^2 = -10$ — Complete the square for $x^2 + 8x$.
$(x + 4)^2 = -10 + 16$ — Add 4^2 to both sides.
$(x + 4)^2 = 6$
$(x + 4) = \pm\sqrt{6}$ — Take square roots of both sides.
$x = -4 \pm \sqrt{6}$ — Subtract 4 from both sides.
So the solutions are
$x = -4 + \sqrt{6}$ and $x = -4 - \sqrt{6}$. — Leave your answer in surd form.

Example 7

Solve the equation $2x^2 - 8x + 7 = 0$. Give your answers in surd form.

$2x^2 - 8x + 7 = 0$

$x^2 - 4x + \frac{7}{2} = 0$

$x^2 - 4x = -\frac{7}{2}$

$(x - 2)^2 - 2^2 = -\frac{7}{2}$ — Complete the square for $x^2 - 4x$.

$(x - 2)^2 = -\frac{7}{2} + 4$ — Add 2^2 to both sides.

$(x - 2)^2 = \frac{1}{2}$

$x - 2 = \pm\sqrt{\frac{1}{2}}$ — Take square roots of both sides.

$x = 2 \pm \frac{1}{\sqrt{2}}$ — Add 2 to both sides.

So the roots are

$x = 2 + \frac{1}{\sqrt{2}}$ and $x = 2 - \frac{1}{\sqrt{2}}$

Problem-solving

This is an **equation** so you can divide every term by the same constant. Divide by 2 to get x^2 on its own. The right-hand side is 0 so it is unchanged.

Online Use your calculator to check solutions to quadratic equations quickly.

Exercise 2D

1 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $x^2 + 6x + 1 = 0$ **b** $x^2 + 12x + 3 = 0$ **c** $x^2 + 4x - 2 = 0$ **d** $x^2 - 10x = 5$

2 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $2x^2 + 6x - 3 = 0$ **b** $5x^2 + 8x - 2 = 0$ **c** $4x^2 - x - 8 = 0$ **d** $15 - 6x - 2x^2 = 0$

E **3** $x^2 - 14x + 1 = (x + p)^2 + q$, where p and q are constants.

a Find the values of p and q. **(2 marks)**

b Using your answer to part **a**, or otherwise, show that the solutions to the equation $x^2 - 14x + 1 = 0$ can be written in the form $r \pm s\sqrt{3}$, where r and s are constants to be found. **(2 marks)**

E/P **4** By completing the square, show that the solutions to the equation $x^2 + 2bx + c = 0$ are given by the formula $x = -b \pm \sqrt{b^2 - c}$. **(4 marks)**

Problem-solving

Follow the same steps as you would if the coefficients were numbers.

Challenge

a Show that the solutions to the equation

$ax^2 + 2bx + c = 0$ are given by $x = -\frac{b}{a} \pm \sqrt{\frac{b^2 - ac}{a^2}}$.

b Hence, or otherwise, show that the solutions to the equation $ax^2 + bx + c = 0$ can be written as

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hint Start by dividing the whole equation by a.

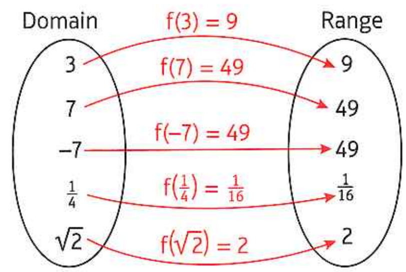
Links You can use this method to prove the quadratic formula. → Section 7.4

2.3 Functions

A function is a mathematical relationship that maps each value of a set of inputs to a single output. The notation f(x) is used to represent a function of x.

- **The set of possible inputs for a function is called the domain.**
- **The set of possible outputs of a function is called the range.**

This diagram shows how the function $f(x) = x^2$ maps five values in its domain to values in its range.



- **The roots of a function are the values of x for which $f(x) = 0$.**

Example 8

The functions f and g are given by $f(x) = 2x - 10$ and $g(x) = x^2 - 9$, $x \in \mathbb{R}$.

a Find the values of f(5) and g(10).

b Find the value of x for which $f(x) = g(x)$.

Notation If the input of a function, x, can be any real number the domain can be written as $x \in \mathbb{R}$.

The symbol $\in$ means 'is a member of' and the symbol $\mathbb{R}$ represents the real numbers.

a $f(5) = 2(5) - 10 = 10 - 10 = 0$ — To find f(5), substitute $x = 5$ into the function f(x).

$g(10) = (10)^2 - 9 = 100 - 9 = 91$

b $f(x) = g(x)$ — Set f(x) equal to g(x) and solve for x.

$2x - 10 = x^2 - 9$

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

$x = 1$

Example 9

The function f is defined as $f(x) = x^2 + 6x - 5$, $x \in \mathbb{R}$.

a Write f(x) in the form $(x + p)^2 + q$.

b Hence, or otherwise, find the roots of f(x), leaving your answers in surd form.

c Write down the minimum value of f(x), and state the value of x for which it occurs.

a $f(x) = x^2 + 6x - 5$

$= (x + 3)^2 - 9 - 5$

$= (x + 3)^2 - 14$

Complete the square for $x^2 + 6x$ and then simplify the expression.

b $f(x) = 0$ — To find the root(s) of a function, set it equal to zero.

$(x + 3)^2 - 14 = 0$

$(x + 3)^2 = 14$ — You can solve this equation directly. Remember to write ± when you take square roots of both sides.

$x + 3 = \pm\sqrt{14}$

$x = -3 \pm \sqrt{14}$

f(x) has two roots:

$-3 + \sqrt{14}$ and $-3 - \sqrt{14}$.

c $(x + 3)^2 \geqslant 0$
So the minimum value of $f(x)$ is -14.
This occurs when $(x + 3)^2 = 0$,
so when $x = -3$

A squared value must be greater than or equal to 0.

$(x + 3)^2 \geqslant 0$ so $(x + 3)^2 - 14 \geqslant -14$

Example 10

Find the roots of the function $f(x) = x^6 + 7x^3 - 8$, $x \in \mathbb{R}$.

$f(x) = 0$
$x^6 + 7x^3 - 8 = 0$
$(x^3)^2 + 7(x^3) - 8 = 0$
$(x^3 - 1)(x^3 + 8) = 0$
So $x^3 = 1$ or $x^3 = -8$
$x^3 = 1 \Rightarrow x = 1$
$x^3 = -8 \Rightarrow x = -2$
The roots of $f(x)$ are 1 and -2.

Alternatively, let $u = x^3$.
$f(x) = x^6 + 7x^3 - 8$
$= (x^3)^2 + 7(x^3) - 8$
$= u^2 + 7u - 8$
$= (u - 1)(u + 8)$
So when $f(x) = 0$, $u = 1$ or $u = -8$.
If $u = 1 \Rightarrow x^3 = 1 \Rightarrow x = 1$
If $u = -8 \Rightarrow x^3 = -8 \Rightarrow x = -2$
The roots of $f(x)$ are 1 and -2.

Problem-solving $f(x)$ can be written as a function of a function. The only powers of x in $f(x)$ are 6, 3 and 0 so you can write it as a quadratic function of x^3.

Treat x^3 as a single variable and factorise.

Solve the quadratic equation to find two values for x^3, then find the corresponding values of x.

You can simplify this working with a substitution.

Replace x^3 with u and solve the quadratic equation in u.

Watch out The solutions to the quadratic equation will be values of u. Convert back to values of x using your substitution.

Exercise 2E

1 Using the functions $f(x) = 5x + 3$, $g(x) = x^2 - 2$ and $h(x) = \sqrt{x+1}$, find the values of:

a $f(1)$ **b** $g(3)$ **c** $h(8)$ **d** $f(1.5)$ **e** $g(\sqrt{2})$

f $h(-1)$ **g** $f(4) + g(2)$ **h** $f(0) + g(0) + h(0)$ **i** $\dfrac{g(4)}{h(3)}$

(P) **2** The function $f(x)$ is defined by $f(x) = x^2 - 2x$, $x \in \mathbb{R}$. Given that $f(a) = 8$, find two possible values for a.

Problem-solving Substitute $x = a$ into the function and set the resulting expression equal to 8.

3 Find all of the roots of the following functions:

a $f(x) = 10 - 15x$ **b** $g(x) = (x + 9)(x - 2)$ **c** $h(x) = x^2 + 6x - 40$

d $j(x) = 144 - x^2$ **e** $k(x) = x(x + 5)(x + 7)$ **f** $m(x) = x^3 + 5x^2 - 24x$

4 The functions p and q are given by $p(x) = x^2 - 3x$ and $q(x) = 2x - 6$, $x \in \mathbb{R}$.
Find the two values of x for which $p(x) = q(x)$.

5 The functions f and g are given by $f(x) = 2x^3 + 30x$ and $g(x) = 17x^2$, $x \in \mathbb{R}$.
Find the three values of x for which $f(x) = g(x)$.

(E) 6 The function f is defined as $f(x) = x^2 - 2x + 2$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(x + p)^2 + q$, where p and q are constants to be found. **(2 marks)**

b Hence, or otherwise, explain why $f(x) > 0$ for all values of x, and find the minimum value of $f(x)$. **(1 mark)**

7 Find all roots of the following functions:

a $f(x) = x^6 + 9x^3 + 8$ **b** $g(x) = x^4 - 12x^2 + 32$

c $h(x) = 27x^6 + 26x^3 - 1$ **d** $j(x) = 32x^{10} - 33x^5 + 1$

e $k(x) = x - 7\sqrt{x} + 10$ **f** $m(x) = 2x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 12$

Hint The function in part **b** has four roots.

(E/P) 8 The function f is defined as $f(x) = 3^{2x} - 28(3^x) + 27$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(3^x - a)(3^x - b)$, where a and b are real constants. **(2 marks)**

b Hence find the two roots of $f(x)$. **(2 marks)**

Problem-solving
Consider $f(x)$ as a function of a function.

2.4 Quadratic graphs

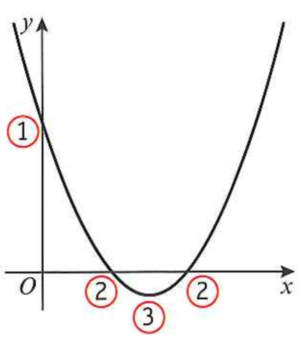
When $f(x) = ax^2 + bx + c$, the graph of $y = f(x)$ has a curved shape called a parabola.

You can sketch a quadratic graph by identifying key features.

The coefficient of x^2 determines the overall shape of the graph.

When a is positive the parabola will have this shape: $\cup$

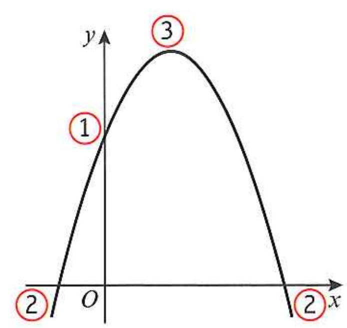
When a is negative the parabola will have this shape: $\cap$



(1) The graph crosses the y-axis when $x = 0$. The y-coordinate is equal to c.

(2) The graph crosses the x-axis when $y = 0$. The x-coordinates are roots of the function $f(x)$.

(3) Quadratic graphs have one turning point. This can be a minimum or a maximum. Since a parabola is symmetrical, the turning point and line of symmetry are half-way between the two roots.



- **You can find the coordinates of the turning point of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.**

Links The graph of $y = a(x + p)^2 + q$ is a translation of the graph of $y = ax^2$ by $\begin{pmatrix} -p \\ q \end{pmatrix}$. → Section 4.5

Example 11

Sketch the graph of $y = x^2 - 5x + 4$, and find the coordinates of its turning point.

As $a = 1$ is positive, the graph has a $\vee$ shape and a minimum point.

When $x = 0$, $y = 4$, so the graph crosses the y-axis at (0, 4).

When $y = 0$,

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$x = 1$ or $x = 4$, so the graph crosses the x-axis at (1, 0) and (4, 0).

Completing the square:

$$x^2 - 5x + 4 = \left(x - \tfrac{5}{2}\right)^2 - \tfrac{25}{4} + 4$$

$$= \left(x - \tfrac{5}{2}\right)^2 - \tfrac{9}{4}$$

So the minimum point has coordinates $\left(\tfrac{5}{2}, -\tfrac{9}{4}\right)$.

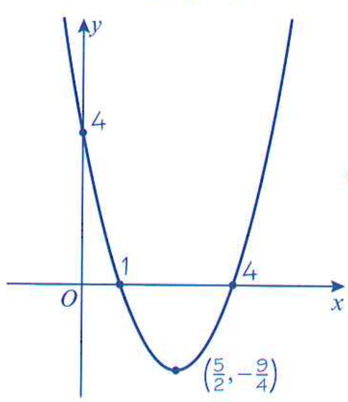
Alternatively, the minimum occurs when x is half-way between 1 and 4,

$$\text{so } x = \frac{1 + 4}{2} = \frac{5}{2}$$

$$y = \left(\tfrac{5}{2}\right)^2 - 5 \times \left(\tfrac{5}{2}\right) + 4 = -\tfrac{9}{4}$$

so the minimum has coordinates $\left(\tfrac{5}{2}, -\tfrac{9}{4}\right)$.

The sketch of the graph is:



Use the coefficient of x^2 to determine the general shape of the graph.

This example factorises, but you may need to use the quadratic formula or complete the square.

Complete the square to find the coordinates of the turning point.

Watch out If you use symmetry to find the x-coordinate of the minimum point, you need to substitute this value into the equation to find the y-coordinate of the minimum point.

You could use a graphic calculator or substitute some values to check your sketch.

When $x = 5$, $y = 5^2 - 5 \times 5 + 4 = 4$.

Online Explore how the graph of $y = (x + p)^2 + q$ changes as the values of p and q change using GeoGebra.

Example 12

Sketch the graph of $y = 4x - 2x^2 - 3$. Find the coordinates of its turning point and write down the equation of its line of symmetry.

As $a = -2$ is negative, the graph has a $\wedge$ shape and a maximum point.

When $x = 0$, $y = -3$, so the graph crosses the y-axis at $(0, -3)$.

When $y = 0$,

$-2x^2 + 4x - 3 = 0$

Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-3)}}{2 \times (-2)}$$

$$x = \frac{-4 \pm \sqrt{-8}}{-4}$$

There are no real solutions, so the graph does not cross the x-axis.

Completing the square:

$-2x^2 + 4x - 3$

$= -2(x^2 - 2x) - 3$

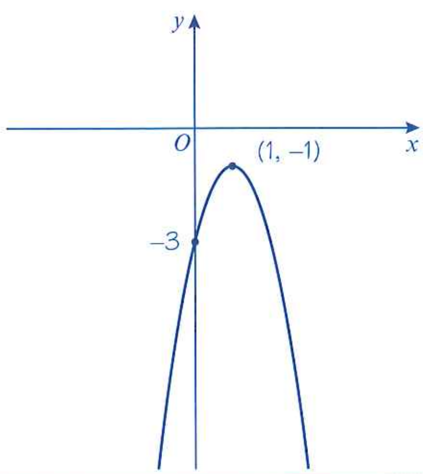
$= -2((x - 1)^2 - 1) - 3$

$= -2(x - 1)^2 + 2 - 3$

$= -2(x - 1)^2 - 1$

So the maximum point has coordinates $(1, -1)$.

The line of symmetry is vertical and goes through the maximum point. It has the equation $x = 1$.



It's easier to see that $a < 0$ if you write the equation in the form $y = -2x^2 + 4x - 3$.

$a = -2$, $b = 4$ and $c = -3$

You would need to square root a negative number to evaluate this expression. Therefore this equation has no real solutions.

The coefficient of x^2 is -2 so take out a factor of -2

Watch out A sketch graph does not need to be plotted exactly or drawn to scale. However you should:

- draw a smooth curve by hand
- identify any relevant key points (such as intercepts and turning points)
- label your axes.

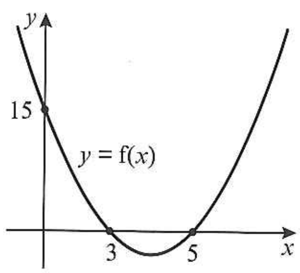
Exercise 2F

1 Sketch the graphs of the following equations. For each graph, show the coordinates of the point(s) where the graph crosses the coordinate axes, and write down the coordinate of the turning point and the equation of the line of symmetry.

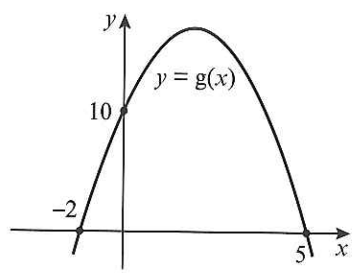
a $y = x^2 - 6x + 8$ **b** $y = x^2 + 2x - 15$ **c** $y = 25 - x^2$ **d** $y = x^2 + 3x + 2$
e $y = -x^2 + 6x + 7$ **f** $y = 2x^2 + 4x + 10$ **g** $y = 2x^2 + 7x - 15$ **h** $y = 6x^2 - 19x + 10$
i $y = 4 - 7x - 2x^2$ **j** $y = 0.5x^2 + 0.2x + 0.02$

(P) 2 These sketches are graphs of quadratic functions of the form $ax^2 + bx + c$. Find the values of a, b and c for each function.

a



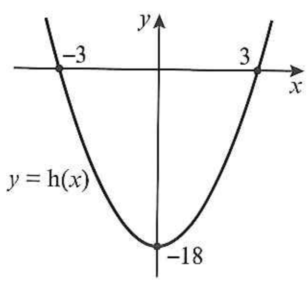
b



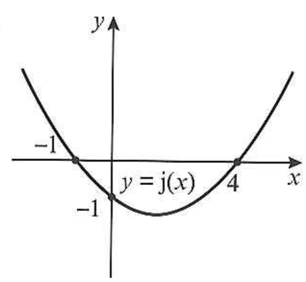
Problem-solving

Check your answers by substituting values into the function. In part **c** the graph passes through (0, −18), so h(0) should be −18.

c



d



(E/P) 3 The graph of $y = ax^2 + bx + c$ has a minimum at $(5, -3)$ and passes through $(4, 0)$. Find the values of a, b and c. **(3 marks)**

2.5 The discriminant

If you square any real number, the result is greater than or equal to 0. This means that if y is negative, $\sqrt{y}$ cannot be a real number. Look at the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

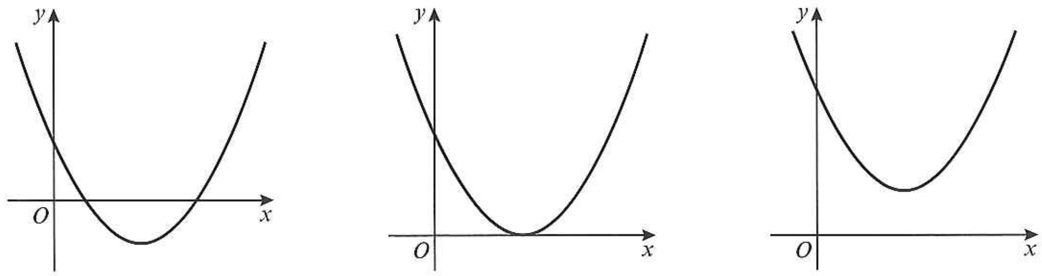
If the value under the square root sign is negative, x cannot be a real number and there are no real solutions. If the value under the square root is equal to 0, both solutions will be the same.

- **For the quadratic function $f(x) = ax^2 + bx + c$, the expression $b^2 - 4ac$ is called the discriminant. The value of the discriminant shows how many roots f(x) has:**
  - **If $b^2 - 4ac > 0$ then f(x) has two distinct real roots.**
  - **If $b^2 - 4ac = 0$ then f(x) has one repeated root.**
  - **If $b^2 - 4ac < 0$ then f(x) has no real roots.**

You can use the discriminant to check the shape of sketch graphs.

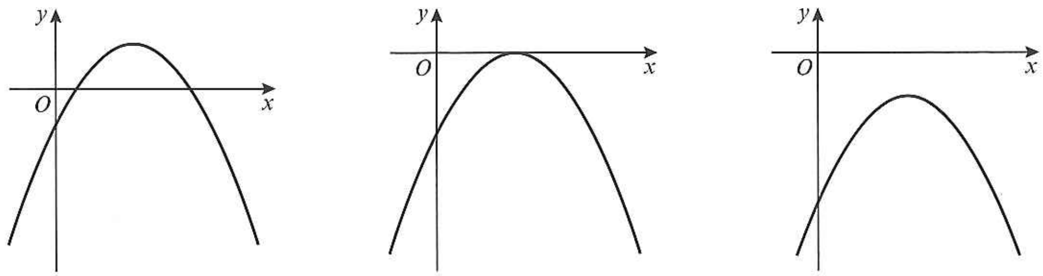
Below are some graphs of $y = f(x)$ where $f(x) = ax^2 + bx + c$.

$a > 0$



$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Two distinct real roots	One repeated root	No real roots

$a < 0$



Example 13

Find the values of k for which $f(x) = x^2 + kx + 9$ has equal roots.

$x^2 + kx + 9 = 0$
Here $a = 1$, $b = k$ and $c = 9$
For equal roots, $b^2 - 4ac = 0$
$k^2 - 4 \times 1 \times 9 = 0$
$k^2 - 36 = 0$
$k^2 = 36$
so $k = \pm 6$

Problem-solving

Use the condition given in the question to write a statement about the discriminant.

Substitute for a, b and c to get an equation with one unknown.

Solve to find the values of k.

Example 14

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

$x^2 + 4x + k = 0$
Here $a = 1$, $b = 4$ and $c = k$.
For two real solutions, $b^2 - 4ac > 0$
$4^2 - 4 \times 1 \times k > 0$
$16 - 4k > 0$
$16 > 4k$
$4 > k$
So $k < 4$

This statement involves an inequality, so your answer will also be an inequality.

For any value of k less than 4, the equation will have 2 distinct real solutions.

Online Explore how the value of the discriminant changes with k using GeoGebra.

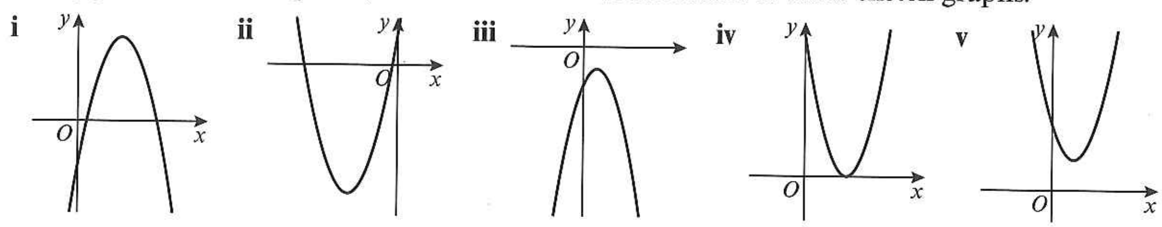
Exercise 2G

1 a Calculate the value of the discriminant for each of these five functions:

i $f(x) = x^2 + 8x + 3$ ii $g(x) = 2x^2 - 3x + 4$ iii $h(x) = -x^2 + 7x - 3$

iv $j(x) = x^2 - 8x + 16$ v $k(x) = 2x - 3x^2 - 4$

b Using your answers to part **a**, match the same five functions to these sketch graphs.



E/P 2 Find the values of k for which $x^2 + 6x + k = 0$ has two real solutions. **(2 marks)**

E/P 3 Find the value of t for which $2x^2 - 3x + t = 0$ has exactly one solution. **(2 marks)**

E/P 4 Given that the function $f(x) = sx^2 + 8x + s$ has equal roots, find the value of the positive constant s. **(2 marks)**

E/P 5 Find the range of values of k for which $3x^2 - 4x + k = 0$ has no real solutions. **(2 marks)**

E/P 6 The function $g(x) = x^2 + 3px + (14p - 3)$, where p is an integer, has two equal roots.

a Find the value of p. **(2 marks)**

b For this value of p, solve the equation $x^2 + 3px + (14p - 3) = 0$. **(2 marks)**

E/P 7 $h(x) = 2x^2 + (k + 4)x + k$, where k is a real constant.

a Find the discriminant of $h(x)$ in terms of k. **(3 marks)**

b Hence or otherwise, prove that $h(x)$ has two distinct real roots for all values of k. **(3 marks)**

Problem-solving

If a question part says 'hence or otherwise' it is usually easier to use your answer to the previous question part.

Challenge

a Prove that, if the values of a and c are given and non-zero, it is always possible to choose a value of b so that $f(x) = ax^2 + bx + c$ has distinct real roots.

b Is it always possible to choose a value of b so that $f(x)$ has equal roots? Explain your answer.

2.6 Modelling with quadratics

A **mathematical model** is a mathematical description of a real-life situation. Mathematical models use the language and tools of mathematics to represent and explore real-life patterns and relationships, and to predict what is going to happen next.

Models can be simple or complicated, and their results can be approximate or exact. Sometimes a model is only valid under certain circumstances, or for a limited range of inputs. You will learn more about how models involve simplifications and assumptions in Statistics and Mechanics.

Quadratic functions can be used to model and explore a range of practical contexts, including projectile motion.

Example 15

A spear is thrown over level ground from the top of a tower.
The height, in metres, of the spear above the ground after t seconds is modelled by the function:

$$h(t) = 12.25 + 14.7t - 4.9t^2, \quad t \geqslant 0$$

a Interpret the meaning of the constant term 12.25 in the model.

b After how many seconds does the spear hit the ground?

c Write h(t) in the form $A - B(t - C)^2$, where A, B and C are constants to be found.

d Using your answer to part **c** or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached.

a The tower is 12.25 m tall, since this is the height at time 0.

b When the spear hits the ground, the height is equal to 0.

$$12.25 + 14.7t - 4.9t^2 = 0$$

Using the formula, where $a = -4.9$, $b = 14.7$ and $c = 12.25$,

$$t = \frac{-14.7 \pm \sqrt{14.7^2 - 4(-4.9)(12.25)}}{(2 \times -4.9)}$$

$$t = \frac{-14.7 \pm \sqrt{456.19}}{-9.8}$$

$t = -0.679$ or $t = 3.68$ (to 3 s.f.)

As $t \geqslant 0$, $t = 3.68$ seconds (to 3 s.f.).

c $12.25 + 14.7t - 4.9t^2$

$= -4.9(t^2 - 3t) + 12.25$

$= -4.9((t - 1.5)^2 - 2.25) + 12.25$

$= -4.9((t - 1.5)^2 + 11.025 + 12.25)$

$= 23.275 - 4.9(t - 1.5)^2$

So $A = 23.275$, $B = 4.9$ and $C = 1.5$.

d The maximum height of the spear is 23.275 metres, 1.5 seconds after the spear is thrown.

Problem-solving

Read the question carefully to work out the meaning of the constant term in the **context of the model**. Here, $t = 0$ is the time the spear is thrown.

To solve a quadratic, factorise, use the quadratic formula, or complete the square.

Give any non-exact numerical answers correct to 3 significant figures unless specified otherwise.

Always interpret your answers in the context of the model. t is the time after the spear was thrown so it must be positive.

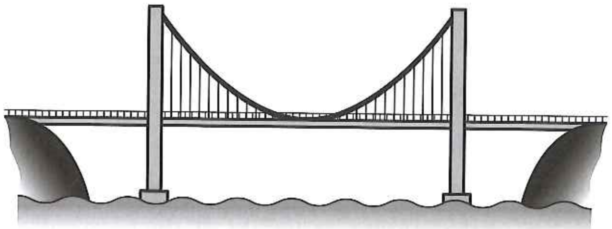
$4.9(t - 1.5)^2$ must be positive or 0, so $h(t) \leqslant 23.275$ for all possible values of t.

The turning point of the graph of this function would be at (1.5, 23.275). You may find it helpful to draw a sketch of the function when working through modelling questions.

Online Explore the trajectory of the spear using GeoGebra.

Exercise 2H

E/P **1** The diagram shows a section of a suspension bridge carrying a road over water.



Problem-solving

For part **a**, make sure your answer is in the context of the model.

The height of the cables above water level in metres can be modelled by the function $h(x) = 0.000\,12x^2 + 200$, where x is the displacement in metres from the centre of the bridge.

a Interpret the meaning of the constant term 200 in the model. **(1 mark)**

b Use the model to find the two values of x at which the height is 346 m. **(3 marks)**

c Given that the towers at each end are 346 m tall, use your answer to part **b** to calculate the length of the bridge to the nearest metre. **(1 mark)**

E/P **2** A car manufacturer uses a model to predict the fuel consumption, y miles per gallon (mpg), for a specific model of car travelling at a speed of x mph.

$$y = -0.01x^2 + 0.975x + 16, \quad x > 0$$

a Use the model to find two speeds at which the car has a fuel consumption of 32.5 mpg. **(3 marks)**

b Rewrite y in the form $A - B(x - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b**, find the speed at which the car has the greatest fuel efficiency. **(1 mark)**

d Use the model to calculate the fuel consumption of a car travelling at 120 mph. Comment on the validity of using this model for very high speeds. **(2 marks)**

E/P **3** A fertiliser company uses a model to determine how the amount of fertiliser used, f kilograms per hectare, affects the grain yield g, measured in tonnes per hectare.

$$g = 6 + 0.03f - 0.000\,06f^2$$

a According to the model, how much grain would each hectare yield without any fertiliser? **(1 mark)**

b One farmer currently uses 20 kilograms of fertiliser per hectare. How much more fertiliser would he need to use to increase his grain yield by 1 tonne per hectare? **(4 marks)**

E/P **4** A football stadium has 25 000 seats. The football club know from past experience that they will sell only 10 000 tickets if each ticket costs £30. They also expect to sell 1000 more tickets every time the price goes down by £1.

a The number of tickets sold t can be modelled by the linear equation $t = M - 1000p$, where £p is the price of each ticket and M is a constant. Find the value of M. **(1 mark)**

The total revenue, £r, can be calculated by multiplying the number of tickets sold by the price of each ticket. This can be written as $r = p(M - 1000p)$.

b Rearrange r into the form $A - B(p - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b** or otherwise, work out how much the football club should charge for each ticket if they want to make the maximum amount of money. **(2 marks)**

Challenge

Accident investigators are studying the stopping distance of a particular car.
When the car is travelling at 20 mph, its stopping distance is 6 feet.
When the car is travelling at 30 mph, its stopping distance is 14 feet.
When the car is travelling at 40 mph, its stopping distance is 24 feet.

The investigators suggest that the stopping distance in feet, d, is a quadratic function of the speed in miles per hour, s.

a Given that $d(s) = as^2 + bs + c$, find the values of the constants a, b and c.

b At an accident scene a car has left behind a skid that is 20 feet long.
Use your model to calculate the speed that this car was going at before the accident.

Hint Start by setting up three simultaneous equations. Combine two different pairs of equations to eliminate c. Use the results to find the values of a and b first.

Mixed exercise 2

1 Solve the following equations without a calculator. Leave your answers in surd form where necessary.

a $y^2 + 3y + 2 = 0$ **b** $3x^2 + 13x - 10 = 0$ **c** $5x^2 - 10x = 4x + 3$ **d** $(2x - 5)^2 = 7$

2 Sketch graphs of the following equations:

a $y = x^2 + 5x + 4$ **b** $y = 2x^2 + x - 3$ **c** $y = 6 - 10x - 4x^2$ **d** $y = 15x - 2x^2$

(E) **3** $f(x) = x^2 + 3x - 5$ and $g(x) = 4x + k$, where k is a constant.

a Given that $f(3) = g(3)$, find the value of k. **(3 marks)**

b Find the values of x for which $f(x) = g(x)$. **(3 marks)**

4 Solve the following equations, giving your answers correct to 3 significant figures:

a $k^2 + 11k - 1 = 0$ **b** $2t^2 - 5t + 1 = 0$ **c** $10 - x - x^2 = 7$ **d** $(3x - 1)^2 = 3 - x^2$

5 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:

a $x^2 + 12x - 9$ **b** $5x^2 - 40x + 13$ **c** $8x - 2x^2$ **d** $3x^2 - (x + 1)^2$

(E) **6** Find the value k for which the equation $5x^2 - 2x + k = 0$ has exactly one solution. **(2 marks)**

(E) 7 Given that for all values of x:

$$3x^2 + 12x + 5 = p(x + q)^2 + r$$

a find the values of p, q and r. **(3 marks)**

b Hence solve the equation $3x^2 + 12x + 5 = 0$. **(2 marks)**

(E/P) 8 The function f is defined as $f(x) = 2^{2x} - 20(2^x) + 64, x \in \mathbb{R}$.

a Write f(x) in the form $(2^x - a)(2^x - b)$, where a and b are real constants. **(2 marks)**

b Hence find the two roots of f(x). **(2 marks)**

9 Find, as surds, the roots of the equation:

$$2(x + 1)(x - 4) - (x - 2)^2 = 0.$$

10 Use algebra to solve $(x - 1)(x + 2) = 18$.

(E/P) 11 A diver launches herself off a springboard. The height of the diver, in metres, above the pool t seconds after launch can be modelled by the following function:

$$h(t) = 5t - 10t^2 + 10, t \geqslant 0$$

a How high is the springboard above the water? **(1 mark)**

b Use the model to find the time at which the diver hits the water. **(3 marks)**

c Rearrange h(t) into the form $A - B(t - C)^2$ and give the values of the constants A, B and C. **(3 marks)**

d Using your answer to part **c** or otherwise, find the maximum height of the diver, and the time at which this maximum height is reached. **(2 marks)**

(E/P) 12 For this question, $f(x) = 4kx^2 + (4k + 2)x + 1$, where k is a real constant.

a Find the discriminant of f(x) in terms of k. **(3 marks)**

b By simplifying your answer to part **a** or otherwise, prove that f(x) has two distinct real roots for all non-zero values of k. **(2 marks)**

c Explain why f(x) cannot have two distinct real roots when $k = 0$. **(1 mark)**

(E/P) 13 Find all of the roots of the function $r(x) = x^8 - 17x^4 + 16$. **(5 marks)**

(E/P) 14 Lynn is selling cushions as part of an enterprise project. On her first attempt, she sold 80 cushions at the cost of £15 each. She hopes to sell more cushions next time. Her adviser suggests that she can expect to sell 10 more cushions for every £1 that she lowers the price.

a The number of cushions sold c can be modelled by the equation $c = 230 - Hp$, where £p is the price of each cushion and H is a constant. Determine the value of H. **(1 mark)**

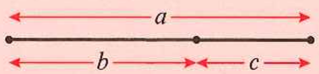
To model her total revenue, £r, Lynn multiplies the number of cushions sold by the price of each cushion. She writes this as $r = p(230 - Hp)$.

b Rearrange r into the form $A - B(p - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b** or otherwise, show that Lynn can increase her revenue by £122.50 through lowering her prices, and state the optimum selling price of a cushion. **(2 marks)**

Challenge

a The ratio of the lengths $a:b$ in this line is the same as the ratio of the lengths $b:c$.



Show that this ratio is $\frac{1+\sqrt{5}}{2}:1$.

b Show also that the infinite square root

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}} = \frac{1+\sqrt{5}}{2}$$

Summary of key points

1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a^2}\right)$

5 The set of possible inputs for a function is called the **domain**.

The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then a quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then a quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then a quadratic function has no real roots

9 Quadratics can be used to model real-life situations.